

# A class of marked invariant subspaces with an application to algebraic Riccati equations

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## Abstract

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**Abstract:** Invariant subspaces of a matrix  $A$  are considered which are obtained by truncation of a Jordan basis of a generalized eigenspace of  $A$ . We characterize those subspaces which are independent of the choice of the Jordan basis. An application to Hamilton matrices and algebraic Riccati equations is given.

# 1 Invariant subspaces

Let  $\lambda$  be an eigenvalue of a complex  $n \times n$  matrix  $A$  and let

$$E_\lambda(A) = \text{Ker}(A - \lambda I)^n$$

be the corresponding generalized eigenspace. Suppose  $\dim E_\lambda(A) = k$ . If

$$(s - \lambda)^{t_1}, \dots, (s - \lambda)^{t_k}, \quad t_1 \leq \dots \leq t_k,$$

are the corresponding elementary divisors then  $E_\lambda(A)$  is a direct sum of  $t_i$ -dimensional cyclic subspaces, i.e.

$$E_\lambda(A) = K_1 \oplus \dots \oplus K_k$$

with

$$K_i = \text{span}\{u_i, (A - \lambda I)u_i, \dots, (A - \lambda I)^{t_i-1}u_i\}, \quad (1.1)$$

and  $(A - \lambda I)^{t_i}u_i = 0$ ,  $i = 1, \dots, k$ . We call

$$U = (u_1, \dots, u_k) \quad (1.2)$$

a tuple of *generators* of  $E_\lambda(A)$ . From a given  $U$  one can construct  $A$ -invariant subspaces in the following way. Let  $r = (r_1, \dots, r_k)$  be such that

$$0 \leq r_i < t_i, \quad i = 1, \dots, k. \quad (1.3)$$

We set

$$W_{r_i}(U) = \text{span}\{(A - \lambda I)^{r_i}u_i, (A - \lambda I)^{r_i+1}u_i, \dots, (A - \lambda I)^{t_i-1}u_i\} \quad (1.4)$$

and

$$W(r, U) = W_{r_1}(U) \oplus \dots \oplus W_{r_k}(U). \quad (1.5)$$

The construction of invariant subspaces of the form  $W(r, U)$  is a standard procedure in linear algebra and systems theory (see e.g. [8], [6, p.61], [5], [10], [2, p.28]).

If  $U$  and  $\tilde{U}$  are two different tuples of generators of  $E_\lambda(A)$  then the restrictions of  $A$  to  $W(r, U)$  and  $W(r, \tilde{U})$  have the same elementary divisors, namely  $(s - \lambda)^{t_i - r_i}$ ,  $i = 1, \dots, k$ . However, in general, the subspaces  $W(r, U)$  and  $W(r, \tilde{U})$  will be different. Consider the following example with  $k = 2$ ,  $t_1 = 2$ ,  $t_2 = 3$ , and

$$A = \text{diag}(N_2, N_3), \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.6)$$

Let  $e_i$  be a unit vector of  $\mathbb{C}^5$ . Then  $U = \{e_2, e_5\}$  and  $\tilde{U} = \{e_2, e_5 + e_2\}$  are tuples of generators of  $E_0(A) = \text{Ker } A^5 = \mathbb{C}^5$ . If we choose  $r = (1, 0)$ , then  $W(r, U) = \text{span}\{e_1, e_3, e_4, e_5\}$  and  $W(r, \tilde{U}) = \text{span}\{e_1, e_3, e_4 + e_1, e_5 + e_2\}$ . Thus

$$W(r, U) \neq W(r, \tilde{U}). \quad (1.7)$$

On the other hand, if we choose  $r = (1, 2)$ , then

$$W(r, U) = W(r, \tilde{U}). \quad (1.8)$$

It is the purpose of our note to determine those tuples  $r = (r_1, \dots, r_k)$  which have the property that the space  $W(r, U)$  given by (1.4) and (1.5) is independent of the generator tuple  $U$ . The motivation for our study comes from Kucera's survey article [8], which deals with independence of generator tuples in the case of Hamiltonian matrices. In Section 3 we make the connection with [8, p.60] applying a corollary of our main theorem to Hamiltonian matrices and algebraic Riccati equations.

In the sequel we assume that  $\lambda = 0$  is an eigenvalue of  $A$  and we focus on  $E_0(A) = \text{Ker } A^n$ . With each nonzero vector  $v \in E_0(A)$  we associate a *height*  $h(v)$  and an *exponent*  $e(v)$  as follows. Suppose

$$v \in \text{Im } A^q, v \notin \text{Im } A^{q+1}, v \in \text{Ker } A^p, v \notin \text{Ker } A^{p-1}.$$

Then we set  $h(v) = q$  and  $e(v) = p$ . Thus, if  $\lambda = 0$  in (1.1) then the elements of  $U$  in (1.2) satisfy  $e(u_1) = t_1 \leq \dots \leq e(u_k) = t_k$  and  $h(u_i) = 0$ . We define

$$\langle v \rangle = \text{span}\{A^\nu v, \nu \geq 0\}.$$

Then  $\langle v \rangle$  is a cyclic subspace generated by  $v$ , and  $\dim \langle v \rangle = e(v)$ .

## 2 The main result

**Theorem 2.1.** *Let  $A \in \mathbb{C}^{n \times n}$  and let*

$$s^{t_1}, \dots, s^{t_k}, t_1 \leq \dots \leq t_k, \quad (2.1)$$

*be the elementary divisors corresponding to the eigenvalue  $\lambda = 0$ . Let*

$$U = (u_1, \dots, u_k)$$

*be a tuple of generators of  $E_0(A) = \text{Ker } A^n$  such that  $e(u_i) = t_i$ ,  $i = 1, \dots, k$ , and*

$$E_0(A) = \langle u_1 \rangle \oplus \dots \oplus \langle u_k \rangle.$$

Let  $r = (r_1, \dots, r_k)$  be a  $k$ -tuple of integers with  $0 \leq r_i < t_i$ ,  $i = 1, \dots, k$ . Define

$$W(r, U) = \langle A^{r_1} u_1 \rangle \oplus \dots \oplus \langle A^{r_k} u_k \rangle \quad (2.2)$$

and

$$W(r) = (\text{Im } A^{r_1} \cap \text{Ker } A^{t_1-r_1}) + \dots + (\text{Im } A^{r_k} \cap \text{Ker } A^{t_k-r_k}). \quad (2.3)$$

Then the following statements are equivalent:

(i) The  $k$ -tuple  $r = (r_1, \dots, r_k)$  satisfies

$$r_1 \leq \dots \leq r_k, \quad (2.4)$$

and

$$t_1 - r_1 \leq \dots \leq t_k - r_k. \quad (2.5)$$

(ii) The space  $W(r, U)$  is independent of  $U$ .

Moreover, if (2.4) and (2.5) hold then  $W(r, U) = W(r)$ .

Proof. (i)  $\Rightarrow$  (ii). We show that (2.4) and (2.5) imply  $W(r, U) = W(r)$ . Define  $W_{r_s}(U) = \langle A^{r_s} u_s \rangle$  such that (1.5) holds. From

$$W_{r_s}(U) \subseteq \text{Im } A^{r_s} \cap \text{Ker } A^{t_s-r_s}$$

we immediately obtain  $W(r, U) \subseteq W(r)$ . Now let  $x$  be in  $\text{Im } A^{r_s} \cap \text{Ker } A^{t_s-r_s}$ . Then  $x = A^{r_s} y$  for some  $y \in E_0(A)$ , and

$$A^{t_s-r_s} x = A^{t_s} y = 0. \quad (2.6)$$

With respect to the basis

$$\mathcal{B}_U = \{A^{\nu_i} u_i; 0 \leq \nu_i \leq t_i - 1, i = 1, \dots, k\} \quad (2.7)$$

we have

$$y = \sum_{i=1}^k \sum_{\nu_i=0}^{t_i-1} \alpha_{i\nu_i} A^{\nu_i} u_i.$$

Let  $\ell$  be the largest integer such that  $t_\ell \leq t_s$ . Then  $A^{t_s} u_i = 0$  for  $i = 1, \dots, \ell$ . Moreover  $A^{t_s+\nu_i} u_i = 0$  if  $t_s + \nu_i > t_i$ . Therefore

$$A^{t_s} y = \sum_{i>\ell} \sum_{\nu_i=0}^{t_i-t_s-1} \alpha_{i\nu_i} A^{t_s+\nu_i} u_i = 0.$$

Since the vectors of  $\mathcal{B}_U$  are linearly independent we obtain  $\alpha_{i\nu_i} = 0$  for  $i > \ell$  and  $\nu_i = 0, \dots, t_i - t_s - 1$ . Hence

$$y = \sum_{i=1}^{\ell} \sum_{\nu_i=0}^{t_i-1} \alpha_{i\nu_i} A^{\nu_i} u_i + \sum_{i>\ell} \sum_{\nu_i=t_i-t_s}^{t_i-1} \alpha_{i\nu_i} A^{\nu_i} u_i$$

and

$$x = \sum_{i=1}^{\ell} \sum_{\nu_i=0}^{t_i-1} \alpha_{i\nu_i} A^{r_s+\nu_i} u_i + \sum_{i>\ell} \sum_{\nu_i=t_i-t_s}^{t_i-1} \alpha_{i\nu_i} A^{r_s+\nu_i} u_i.$$

Note that  $t_s = \dots = t_\ell$  implies  $r_s = \dots = r_\ell$ . Hence, if  $1 \leq i \leq \ell$  then  $r_i \leq r_s$ , and therefore

$$A^{r_s+\nu_i} u_i \in W_{r_i}(U). \quad (2.8)$$

On the other hand, if  $i > \ell$  then  $t_s - r_s \leq t_i - r_i$ . In that case  $\nu_i \in \{t_i - t_s, \dots, t_i - 1\}$  implies

$$r_s + \nu_i \geq r_s + (t_i - t_s) \geq r_i.$$

Thus, we again have (2.8). Hence  $x \in W(r, U)$  and therefore  $W(r) \subseteq W(r, U)$ .

(ii)  $\Rightarrow$  (i). We assume that  $W(r, U)$  is independent of  $U$ . Let us show first that

$$r_i = r_j \text{ if } t_i = t_j. \quad (2.9)$$

Suppose  $r = (r_1, \dots, r_k)$  is such that  $t_s = t_{s+1}$  and  $r_s \neq r_{s+1}$ , e.g.

$$r_{s+1} < r_s \text{ for some } s \in \{1, \dots, k-1\}. \quad (2.10)$$

Let  $V = (v_1, \dots, v_k)$  be such that  $(v_s, v_{s+1}) = (u_{s+1}, u_s)$ , and  $v_i = u_i$  if  $i \notin \{s, s+1\}$ . Then  $A^{r_{s+1}} u_{s+1} \in W(r, U)$  but  $A^{r_{s+1}} u_{s+1} = A^{r_{s+1}} v_s \notin W(r, V)$ . Therefore the tuples  $U$  and  $V$  contain the same elements, but  $W(r, U) \neq W(r, V)$ .

Now suppose that (2.4) is not satisfied. Then we have (2.10), and

$$A^{r_{s+1}} u_s \notin W(r, U). \quad (2.11)$$

Let  $V = (v_1, \dots, v_k)$  be given by  $v_{s+1} = u_{s+1} + u_s$ , and  $v_i = u_i$ , if  $i \neq s+1$ . Thus  $V$  is a tuple of generators of  $E_0(A)$  with  $e(v_i) = e(u_i)$ . Consider

$$A^{r_{s+1}} u_{s+1} + A^{r_{s+1}} u_s = A^{r_{s+1}} v_{s+1} \in W(r, V).$$

Then  $A^{r_{s+1}} v_{s+1} \notin W(r, U)$ . Otherwise  $A^{r_{s+1}} u_{s+1} \in W(r, U)$  would imply  $A^{r_{s+1}} u_s \in W(r, U)$ , which is a contradiction to (2.11).

Suppose  $r = (r_1, \dots, r_k)$  does not satisfy (2.5). Then  $t_s - r_s > t_{s+1} - r_{s+1}$  for some  $s \in \{1, \dots, k-1\}$ . Because of (2.9) we have  $t_{s+1} \neq t_s$ . Hence  $r_{s+1} - r_s > t_{s+1} - t_s > 0$ , and  $r_s < r_{s+1}$ , and

$$r_s + (t_{s+1} - t_s) < r_{s+1}. \quad (2.12)$$

Because (2.2) it is obvious that (2.12) implies

$$A^{r_s + (t_{s+1} - t_s)} u_{s+1} \notin W(r, U). \quad (2.13)$$

Define  $v_s = u_s + A^{t_{s+1} - t_s} u_{s+1}$ . Then  $e(v_s) = e(u_s) = t_s$ . Therefore

$$V = \{u_1, \dots, u_{s-1}, v_s, u_{s+1}, \dots, u_k\} \quad (2.14)$$

is another tuple of generators of  $E_0(A)$ . Let us show that  $W(r, V) \neq W(r, U)$ . Clearly, the vector  $A^{r_s} v_s$  belongs to  $W(r, V)$ . Suppose

$$A^{r_s} u_s + A^{r_s + (t_{s+1} - t_s)} u_{s+1} = A^{r_s} v_s \in W(r, U).$$

Because of  $A^{r_s} u_s \in W(r, U)$  that would imply

$$A^{r_s + (t_{s+1} - t_s)} u_{s+1} \in W(r, U),$$

which is a contradiction to (2.13). □

Let us consider again Example (1.6). We have  $(t_1, t_2) = (2, 3)$ . In the case of  $r = (1, 0)$  condition (2.4) is violated, which accounts for (1.7). In the case of  $r = (1, 2)$  both (2.4) and (2.5) hold, which ensures (1.8).

In accordance with a definition in [7, p. 83] and [3] the space  $W(r, U)$  is a *marked*  $A$ -invariant subspace of  $E_0(A)$ . That means  $W(r, U)$  has a Jordan basis, in our case

$$\{A^{r_i + \mu_i} u_i; 0 \leq \mu_i \leq t_i - r_i - 1, i = 1, \dots, k\},$$

which can be extended to a Jordan basis of  $E_0(A)$ , namely to  $\mathcal{B}_U$  in (2.7). Let  $\mathcal{M}_r$  be the set of marked subspaces  $M$  of  $E_0(A)$  such that the elementary divisors of the restriction  $A|_M$  are  $s^{t_1 - r_1}, \dots, s^{t_k - r_k}$ . We have noted before that for each tuple of generators  $U$  the corresponding space  $W(r, U)$  is in  $\mathcal{M}_r$ . Suppose (2.4) and (2.5) hold. Then all the spaces  $W(r, U)$  coincide with  $W(r)$  and one might ask whether  $W(r)$  is the only subspaces in  $\mathcal{M}_r$ . In the following we have an example where  $\mathcal{M}_r \supsetneq \{W(r)\}$ . Let  $n = 10$ ,  $k = 2$ , and  $t = (t_1, t_2) = (4, 6)$ , and  $r = (2, 3)$ . Then  $t - r = (2, 3)$ . Hence

the conditions (2.4) and (2.5) are satisfied. Let  $U = (u_1, u_2)$  be a tuple of generators such that  $e(u_1) = 4$  and  $e(u_2) = 6$ . The subspaces

$$M = W(r, U) = W(r) = \langle A^2 u_1 \rangle \oplus \langle A^3 u_2 \rangle$$

and  $\tilde{M} = \langle A u_1 \rangle \oplus \langle A^4 u_2 \rangle$  are marked, the elementary divisors of  $A|_M$  and  $A|_{\tilde{M}}$  are  $s^2, s^3$ . Hence  $\tilde{M} \in \mathcal{M}_r$ , but  $\tilde{M} \neq W(r)$ .

Let  $[m]$  denote the greatest integer of  $m$ . If we assume  $(t_1, \dots, t_k)$  as in (2.1) and take  $r = ([\frac{1}{2}t_1], \dots, [\frac{1}{2}t_k])$  then the conditions (2.4) and (2.5) are satisfied and we note the following corollary of Theorem 2.1.

**Corollary 2.2.** *Let  $A \in \mathbb{C}^{n \times n}$  and  $0 \in \sigma(A)$ . Let  $s^{2m_1}, \dots, s^{2m_k}$ , be the elementary divisors of  $A$  corresponding to  $\lambda = 0$ . If  $U = (u_1, \dots, u_k)$  is a tuple of generators of  $\text{Ker } A^n$  such that  $e(u_i) = 2m_i$ ,  $i = 1, \dots, k$ , then  $e(A^{m_i} u_i) = m_i$  for all  $i$ , and*

$$\langle A^{m_1} u_1 \rangle \oplus \dots \oplus \langle A^{m_k} u_k \rangle = (\text{Im } A^{m_1} \cap \text{Ker } A^{m_1}) + \dots + (\text{Im } A^{m_k} \cap \text{Ker } A^{m_k}).$$

### 3 An application

In this section we apply Corollary 2.2 to the algebraic Riccati equation

$$Q + F^* X + X F - X D X = 0 \quad (3.1)$$

and its associated Hamiltonian matrix

$$H = \begin{pmatrix} F & -D \\ -Q & -F^* \end{pmatrix}. \quad (3.2)$$

Here  $F, D, Q$  are complex  $m \times m$  matrices,  $D$  and  $Q$  are hermitian,  $D \geq 0$ , and the pair  $(F, D)$  is assumed to be controllable. Then (see [8, p.59]) all elementary divisors corresponding to eigenvalues  $i\alpha \in i\mathbb{R}$  have even degree. To fix ideas we assume  $\sigma(H) = \{0\}$ . The subsequent result complements Lemma 3.2.3 of [8, p.60].

**Proposition 3.1.** *Let  $s^{2m_1}, \dots, s^{2m_k}$  be the elementary divisors of  $H$ . Set*

$$W = (\text{Im } H^{m_1} \cap \text{Ker } H^{m_1}) + \dots + (\text{Im } H^{m_k} \cap \text{Ker } H^{m_k}). \quad (3.3)$$

*Then  $W$  is an  $H$ -invariant subspace of  $\mathbb{C}^{2m}$  and  $\dim W = m$ . Let  $Y, Z \in \mathbb{C}^{m \times m}$  be such that the columns of  $\begin{pmatrix} Y \\ Z \end{pmatrix}$  are a basis of  $W$ . Then  $Y$  is nonsingular and  $X = ZY^{-1}$  is the unique hermitian solution of (3.1).*



Proof. Set  $t = (2m_1, \dots, 2m_k)$ . Let  $U = (u_1, \dots, u_k)$  be a tuple of generators of  $E_0(H) = \mathbb{C}^{2m}$ . According to [8] we have

$$W(\tfrac{1}{2}t, U) = \text{span} \begin{pmatrix} I_m \\ X \end{pmatrix},$$

where  $X \in \mathbb{C}^{m \times m}$  is the unique hermitian solution of (3.1). From Corollary 2.2 we know that  $W(\tfrac{1}{2}t, U)$  is independent of the choice of  $U$ . Moreover,  $W(\tfrac{1}{2}t, U) = W$  where  $W$  is given by (3.3). Hence, if  $W = \text{span} \begin{pmatrix} Y \\ Z \end{pmatrix}$  then  $Y$  is nonsingular, and

$$\text{span} \begin{pmatrix} Y \\ Z \end{pmatrix} = \text{span} \begin{pmatrix} I \\ ZY^{-1} \end{pmatrix}$$

implies that  $X = ZY^{-1}$  is the solution of (3.1).  $\square$

## 4 Conclusions

Results of this note can be considered in a module theoretic framework. In a subsequent paper we shall make the connection of Theorem 2.1 with marked subspaces in [4] and with torsion modules over discrete valuation domains in [1].

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